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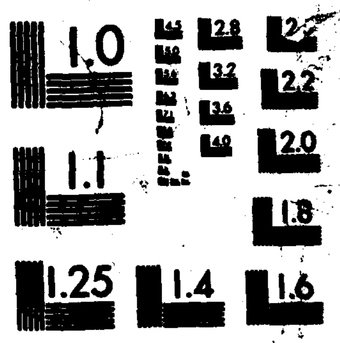

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NWC TP 6462

Addenda to "A Theorem for Physicists in the Theory of Random Variables"

AD-A166 364

by
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JULY 1983

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Naval Weapons Center

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FOREWORD

An article by D. T. Gillespie entitled "A Theorem for Physicists in the Theory of Random Variables" was published in the *American Journal of Physics*, Vol. 51 (June 1983), pp. 520-33. That pedagogically oriented article called attention to a somewhat unorthodox prescription for determining the effects of subjecting random variables to mathematical transformations. The article derived the "random variable transformation (RVT) theorem," and then illustrated the usefulness of that theorem by developing from it a number of important results in statistics and statistical physics. Because of rather stringent page length restrictions in the *American Journal of Physics* (the current average length for an *AJP* article is only about four pages) several appendixes, commentary paragraphs, and illustrative applications had to be deleted from the original journal manuscript. This technical publication contains a compilation of the major items that were deleted, and therefore serves as a supplement to the journal article. This work was done at the Naval Weapons Center during 1982 under Program Element 61152N, Task Area ZR000-01-01, Work Unit 138070.

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(U) *Addenda to "A Theorem for Physicists in the Theory of Random Variables,"* by Daniel T. Gillespie. China Lake, Calif., Naval Weapons Center, July 1983. 17 pp. (NWC TP 6462, publication UNCLASSIFIED.)

(U) This technical publication consists of ten addenda to the article "A Theorem for Physicists in the Theory of Random Variables," which was written by the same author and published in the *American Journal of Physics*, Vol. 51 (June 1983), pp. 520-33. The addenda are in the nature of comments, examples, and appendixes to the original work.

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This paper consists of ten addenda to the article: "A Theorem for Physicists in the Theory of Random Variables" [American Journal of Physics 51 (June, 1983), pp. 520–533]. Numbered references and equations referred to herein are those of that article.

ADDENDUM A. Obtaining Eq. (2).

Fourier's Integral Theorem states that, for any suitably behaved function $f(x)$, we have

$$f(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x') \exp[is(x'-x)]\, dx'ds.$$

If we put $f(x) = \delta(x - x_0)$, then the x'-integration is easily performed, and the result is Eq. (2).

Students of quantum mechanics will recognize Eq. (2) as the orthonormality relation for the eigenfunctions $(2\pi)^{-1/2}\exp(ixs)$ of the operator $-i\partial/\partial s$, with x representing the continuously distributed eigenvalues.

ADDENDUM B. Derivation of Eq. (9).

Although the argument given in the text to deduce the integral formula for $\langle h(X)\rangle$ in Eq. (9) from the definition of $\langle h(X)\rangle$ in Eq. (8) shows that Eq. (9) is indeed quite plausible, that argument cannot be regarded as a rigorous mathematical derivation. In fact, a derivation that is at once rigorous, general, transparent and brief does not seem to exist. As somewhat of a compromise, we state and prove the following theorem, which is essentially a generalization of the so-called weak law of large numbers.

THEOREM : Let $X_1, \ldots, X_n$ be n independent random variables which have a common density function P. Let h be any single-argument function for which the two integrals

$$m \equiv \int_{-\infty}^{\infty} h(x)\, P(x)\, dx \qquad \text{(B1a)}$$

and

$$s^2 \equiv \int_{-\infty}^{\infty} [h(x) - m]^2\, P(x)\, dx \qquad \text{(B1b)}$$

exist. Then, for any given $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathrm{Prob}\left\{ \left| \frac{1}{n}\sum_{i=1}^{n} h(X_i) - m \right| < \varepsilon \right\} = 1 . \qquad \text{(B2)}$$

Since we *define* $h(X)$ to be the random variable whose sample values are $h(x^{(1)})$, $h(x^{(2)})$, ... , where $x^{(1)}$, $x^{(2)}$, ... are the sample values of X, then the above theorem implies that the limit on the right side of Eq. (8) "converges in probability" to the integral on the right side of Eq. (9). Notice that the theorem, as stated here, requires the existence (i.e., finiteness) of the integral in Eq. (B1b). As one might expect, *the theorem is also valid even if the integral in Eq. (B1b) is undefined*; however, the proof in that case is much more complicated. [The more general theorem is due to Khintchine; see K. L. Chung, *A Course in Probability Theory*, Academic Press (1974), pp. 109 and 169.]

To prove this theorem, we begin by formulating and proving a lemma that is essentially a generalization of the well-know Chebyshev Inequality.

LEMMA I (Generalized Chebyshev Inequality): Let $\mathbf{X} \equiv (X_1, \ldots, X_n)$ be a set of n random variables (not necessarily independent) with joint density function Q. Let f be any function of n real variables $\mathbf{x} \equiv (x_1, \ldots, x_n)$ for which the two integrals

$$M \equiv \int_{-\infty}^{\infty} f(\mathbf{x})\, Q(\mathbf{x})\, d\mathbf{x} \qquad \text{and} \qquad S^2 \equiv \int_{-\infty}^{\infty} [f(\mathbf{x}) - M]^2\, Q(\mathbf{x})\, d\mathbf{x} \tag{B3}$$

exist. Then, for any $a > 0$, we have

$$\mathrm{Prob}\{\, |f(\mathbf{X}) - M| \geq aS \,\} \leq a^{-2}\,. \tag{B4}$$

Proof of Lemma I: Let Ω be the set of all points $\mathbf{x}$ for which $|f(\mathbf{x}) - M| \geq aS$. Then

$$\mathrm{Prob}\{\, |f(\mathbf{X}) - M| \geq aS \,\} = \int_{\Omega} Q(\mathbf{x})\, d\mathbf{x} \leq \int_{\Omega} \left| \frac{f(\mathbf{x}) - M}{aS} \right|^2 Q(\mathbf{x})\, d\mathbf{x} \leq \int_{-\infty}^{\infty} \left| \frac{f(\mathbf{x}) - M}{aS} \right|^2 Q(\mathbf{x})\, d\mathbf{x}$$

$$= a^{-2} S^{-2} \int_{-\infty}^{\infty} [f(\mathbf{x}) - M]^2\, Q(\mathbf{x})\, d\mathbf{x} = a^{-2}\,.$$

QED

Now we apply Lemma I to the case in which $X_1, \ldots, X_n$ are n independent random variables with the common density function P, whence $Q(x_1, \ldots, x_n) = P(x_1)P(x_2) \cdots P(x_n)$, and f is the function

$$f(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} h(x_i). \tag{B5}$$

In that case, the integrals M and S^2 defined in Eq. (B3) are given by

$$M = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \left[\frac{1}{n} \sum_{i=1}^{n} h(x_i) \right] \prod_{j=1}^{n} P(x_j), \tag{B6a}$$

and

$$S^2 = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \left[\frac{1}{n} \sum_{i=1}^{n} h(x_i) - M \right]^2 \prod_{j=1}^{n} P(x_j) . \qquad \text{(B6b)}$$

That M and S^2 indeed exist is a consequence of the assumed existence of m and s^2 in Eqs. (B1), together with the following lemma.

LEMMA II: The quantities M and S^2 in Eqs. (B6) are related to m and s^2 in Eqs. (B1) by

$$M = m \quad \text{and} \quad S^2 = s^2/n . \qquad \text{(B7)}$$

Proof of Lemma II: **Letting all integrations run from $-\infty$ to ∞, and using $\int P(x)dx = 1$ together with the definitions of m and s^2 in Eqs. (B1), we evaluate the integrals in Eqs. (B6) thusly:**

$$M = \frac{1}{n} \sum_{i=1}^{n} \int dx_1 \cdots \int dx_n \, h(x_i) \prod_{j=1}^{n} P(x_j) = \frac{1}{n} \sum_{i=1}^{n} \int dx_i \, h(x_i) P(x_i) = \frac{1}{n} (nm) = m .$$

$$S^2 = \frac{1}{n^2} \int dx_1 \cdots \int dx_n \left[\sum_{i=1}^{n} h(x_i) - nm \right]^2 \prod_{j=1}^{n} P(x_j)$$

$$= \frac{1}{n^2} \int dx_1 \cdots \int dx_n \left[\sum_{i=1}^{n} \sum_{k=1}^{n} h(x_i) h(x_k) - 2nm \sum_{i=1}^{n} h(x_i) + n^2 m^2 \right] \prod_{j=1}^{n} P(x_j)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \int dx_1 \cdots \int dx_n \, h^2(x_i) \prod_{j=1}^{n} P(x_j) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k(\neq i)=1}^{n} \int dx_1 \cdots \int dx_n \, h(x_i) h(x_k) \prod_{j=1}^{n} P(x_j)$$

$$- 2n^{-1} m \sum_{i=1}^{n} \int dx_1 \cdots \int dx_n \, h(x_i) \prod_{j=1}^{n} P(x_j) + m^2 \int dx_1 \cdots \int dx_n \prod_{j=1}^{n} P(x_j)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \int dx_i \, h^2(x_i) P(x_i) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k(\neq i)=1}^{n} \int dx_i \, h(x_i) P(x_i) \int dx_k \, h(x_k) P(x_k)$$

$$- 2n^{-1} m \sum_{i=1}^{n} \int dx_i \, h(x_i) P(x_i) + m^2$$

$$= \frac{1}{n^2} n \int dx \, h^2(x) P(x) + \frac{1}{n^2} n(n-1) \, m m - 2n^{-1} m(nm) + m^2$$

$$= \frac{1}{n} \left\{ \int dx \, h^2(x) P(x) - m^2 \right\} = \frac{1}{n} \int dx \, P(x) [h^2(x) - m^2]$$

$$= \frac{1}{n} \int dx \, P(x) [h^2(x) - 2mh(x) + m^2] = \frac{1}{n} \int dx \, P(x) [h(x) - m]^2 \equiv s^2/n . \qquad \text{QED}$$

Substituting for f, M and S from Eqs. (B5) and (B7) into Eq. (B4), we get

$$\text{Prob}\left\{ \left| \frac{1}{n}\sum_{i=1}^{n} h(X_i) - m \right| \geq asn^{-1/2} \right\} \leq a^{-2}. \tag{B8}$$

Letting $\varepsilon = asn^{-1/2}$, so that $a^{-2} = s^2\varepsilon^{-2}n^{-1}$, Eq. (B8) becomes, for any $\varepsilon > 0$,

$$\text{Prob}\left\{ \left| \frac{1}{n}\sum_{i=1}^{n} h(X_i) - m \right| \geq \varepsilon \right\} \leq s^2\varepsilon^{-2}n^{-1},$$

or, since $\text{Prob}\{A < \varepsilon\} \equiv 1 - \text{Prob}\{A \geq \varepsilon\}$,

$$\text{Prob}\left\{ \left| \frac{1}{n}\sum_{i=1}^{n} h(X_i) - m \right| < \varepsilon \right\} \geq 1 - s^2\varepsilon^{-2}n^{-1}. \tag{B9}$$

Now taking the limit of Eq. (B9) as $n \to \infty$, we obtain the desired result in Eq. (B2).

ADDENDUM C. Derivation of Eqs. (23).

The random variables $\bar{X}_n$ and S_n^2 are defined in Eqs. (22) in terms of the random variables $X_1, \ldots, X_n$; these in turn are defined in Eq. (21) in terms of the given random variable X. Since $X_1, \ldots, X_n$ are mutually independent, each having the X density function P, then their joint density function is $P(x_1)P(x_2)\cdots P(x_n)$. Thus, Eq. (18) implies that

$$\langle X_i \rangle = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \, x_i \prod_{k=1}^{n} P(x_k) = \int_{-\infty}^{\infty} x_i P(x_i)\, dx_i \prod_{k=1}^{n}{}' \int_{-\infty}^{\infty} P(x_k)\, dx_k ,$$

where the prime on the product denotes omission of $k = i$. Thus, using Eqs. (6) and (10a),

$$\langle X_i \rangle = \int_{-\infty}^{\infty} x\, P(x)\, dx = \langle X \rangle = \mu. \tag{C1}$$

Similarly, it follows that

$$\langle X_i^2 \rangle = \int_{-\infty}^{\infty} x^2 P(x)\, dx = \langle X^2 \rangle. \tag{C2}$$

Also, for any $i \neq j$, we have

$$\langle X_i X_j \rangle = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \, x_i x_j \prod_{k=1}^{n} P(x_k) = \int_{-\infty}^{\infty} x_i P(x_i)\, dx_i \int_{-\infty}^{\infty} x_j P(x_j)\, dx_j \prod_{k=1}^{n}{}'' \int_{-\infty}^{\infty} P(x_k)\, dx_k ,$$

where the double prime on the product denotes omission of $k = i$ and $k = j$. Thus,

$$\langle X_i X_j \rangle = \left(\int_{-\infty}^{\infty} x\, P(x)\, dx \right)^2 = \mu^2 \quad (i \neq j). \tag{C3}$$

Using the above results and the linearity of the integration operation, we proceed as follows: From Eqs. (22a) and (C1) we have

$$\langle \bar{X}_n \rangle = \langle n^{-1} \sum_{i=1}^{n} X_i \rangle = n^{-1} \sum_{i=1}^{n} \langle X_i \rangle = n^{-1} \sum_{i=1}^{n} \mu ,$$

whence

$$\langle \bar{X}_n \rangle = \mu . \tag{C4}$$

This establishes Eq. (23a). To prove Eqs. (23b) and (23c), we must first calculate $\langle \bar{X}_n{}^2 \rangle$:

$$\langle \bar{X}_n^2 \rangle = \langle (n^{-1} \sum_{i=1}^{n} X_i)(n^{-1} \sum_{j=1}^{n} X_j) \rangle = n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle X_i X_j \rangle$$

$$= n^{-2} \sum_{i=1}^{n} \langle X_i^2 \rangle + n^{-2} \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} \langle X_i X_j \rangle = n^{-2} n \langle X^2 \rangle + n^{-2} n (n-1) \mu^2 ,$$

where the last step uses Eqs. (C2) and (C3). With Eq. (10b), this reduces to

$$\langle \bar{X}_n{}^2 \rangle = n^{-1} \sigma^2 + \mu^2 . \tag{C5}$$

Now using Eqs. (C5) and (C4), we get

$$\langle \bar{X}_n{}^2 \rangle - \langle \bar{X}_n \rangle^2 = n^{-1} \sigma^2 + \mu^2 - \mu^2 = n^{-1} \sigma^2 ,$$

which establishes Eq. (23b). Finally, we have from Eqs. (22b), (C2) and (C5),

$$\langle S_n^2 \rangle = \langle n^{-1} \sum_{i=1}^{n} X_i^2 - \bar{X}_n^2 \rangle = n^{-1} \sum_{i=1}^{n} \langle X_i^2 \rangle - \langle \bar{X}_n^2 \rangle$$

$$= n^{-1} n \langle X^2 \rangle - [n^{-1} \sigma^2 + \mu^2] = \langle X^2 \rangle - \mu^2 - n^{-1} \sigma^2 .$$

Using Eq. (10b) this easily reduces to Eq. (23c).

If we define a new random variable $\Sigma_n{}^2$ by

$$\Sigma_n^2 \equiv \frac{n}{n-1} S_n^2 \equiv \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 , \tag{C6}$$

then Eq. (23c) evidently implies that

$$\langle \Sigma_n{}^2 \rangle = \sigma^2 . \tag{C7}$$

Some writers prefer to define the n-sample mean of X to be $\Sigma_n{}^2$ instead of $S_n{}^2$. Comparing Eqs. (22b) and (C6), we see that as $n \to \infty$ the sample values of *both* $S_n{}^2$ and $\Sigma_n{}^2$ approach σ^2; however, comparing Eqs. (23c) and (C7), we see that only $\Sigma_n{}^2$ has σ^2 as its exact mean. Of course if $n \gg 1$ then the distinction between $S_n{}^2$ and $\Sigma_n{}^2$ is unimportant.

ADDENDUM D. An Alternate Derivation of the RVT Theorem for n = 2, m = 1.

Let the random variables X_1 and X_2 have the joint density function $P(x_1,x_2)$, and let the random variable Y be defined by $Y = f(X_1,X_2)$. To calculate the density function $Q(y)$ of Y, let us assume that the equation $y = f(x_1,x_2)$ can be solved "nicely" for x_2 in terms of x_1 and y; i.e., let us assume that

$$y = f(x_1,x_2) \quad \text{iff} \quad x_2 = h(x_1,y), \tag{D1}$$

where the functions f and h are one-to-one with continuous first derivatives. Then the transformation

$$T: \begin{cases} x_1 = x_1 \\ \\ y = f(x_1,x_2) \end{cases} \tag{D2a}$$

from x_1x_2-space to x_1y-space is one-to-one and differentiable, and so is its inverse,

$$T^{-1}: \begin{cases} x_1 = x_1 \\ \\ x_2 = h(x_1,y) \;. \end{cases} \tag{D2b}$$

Let the infinitesimal area element dx_1dy be the image under T of the infinitesimal area element dx_1dx_2. Then clearly {the probability that a simultaneous sampling of X_1 and Y will yield a point inside dx_1dy} is identical to {the probability that a simultaneous sampling of X_1 and X_2 will yield a point inside dx_1dx_2}. Letting $R(x_1,y)$ denote the joint density function of X_1 and Y, we can express this fact symbolically by

$$R(x_1,y)\,dx_1dy = P(x_1,x_2)\,dx_1dx_2 \,, \tag{D3}$$

where it is understood that (x_1,y) and (x_1,x_2) are related by Eqs. (D2), and that dx_1dx_2 and dx_1dy are related by the corresponding Jacobian formula

$$dx_1dx_2 = \left| \frac{\partial(x_1,x_2)}{\partial(x_1,y)} \right| dx_1dy = |h_y(x_1,y)|\,dx_1dy \,. \tag{D4}$$

Here, h_y denotes the partial derivative of h with respect to y. Substituting Eqs. (D2b) and (D4) into the right side of Eq. (D3), we find that the joint density function of X_1 and Y is

$$R(x_1,y) = P(x_1,\,h(x_1,y))\,|h_y(x_1,y)| \,. \tag{D5}$$

Therefore, according to Eq. (17), the density function of Y alone is

$$Q(y) = \int_{-\infty}^{\infty} dx_1\, P(x_1,h(x_1,y))\,|h_y(x_1,y)| \,. \tag{D6}$$

Notice that this result has been obtained without reference to the Dirac delta function.

Now we observe that Eq. (D6) can also be written

$$Q(y) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} d\xi \; P(x_1, h(x_1,\xi)) \, |h_\xi(x_1,\xi)| \, \delta(\xi - y). \tag{D7}$$

Let us change integration variables in Eq. (D7) from (x_1,ξ) to $(x_1, x_2 \equiv h(x_1,\xi))$. Since $x_2 = h(x_1,\xi)$, then it follows from Eq. (D1), that

$$\xi = f(x_1,x_2)\,;$$

furthermore, just as in Eq. (D4) we have

$$dx_1 dx_2 = \left| \frac{\partial(x_1,x_2)}{\partial(x_1,\xi)} \right| dx_1 d\xi = |h_\xi(x_1,\xi)| \, dx_1 d\xi .$$

Therefore, Eq. (D7) becomes under this change of variable,

$$Q(y) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \, P(x_1,x_2) \, \delta(y - f(x_1,x_2)), \tag{D8}$$

which is precisely Eq. (31) for the case $n = 2$.

Although Eq. (D6) allows $Q(y)$ to be calculated as a one-dimensional integral without having to worry about any Dirac delta function, it describes only one of several possible ways of calculating $Q(y)$, and quite possibly not the easiest way. Eq. (D8) on the other hand offers a variety of ways to proceed: If we change integration variables in Eq. (D8) from (x_1,x_2) to $(x_1, \xi \equiv f(x_1,x_2))$ and then integrate over ξ, we recover Eq. (D6); if we change integration variables in Eq. (D8) from (x_1,x_2) to $(\eta \equiv f(x_1,x_2), x_2)$ and then integrate over η, we obtain a version of Eq. (D6) that would follow by solving $y = f(x_1,x_2)$ for x_1 instead of x_2; or, if we substitute into Eq. (D8) an explicit representation of the delta function, such as that in Eq. (2), we obtain yet another avenue of calculation. Thus, Eq. (D8) is not only more symmetrical, but also more flexible, than Eq. (D6).

Notice that the derivation of the general RVT Theorem given in Sec. III is much more compact than the above derivation of the special case result Eq. (D8).

ADDENDUM E. The Delta Function Change-of-Variable Theorem.

The following theorem is useful when integrating over a delta function when the argument of the delta function is a complicated function of the integration variable.

Theorem: Let h be a differentiable function of x whose only zeros are at $x_1, x_2, \ldots, x_n$, and let $h'(x_i) \neq 0$ for $i = 1, \ldots, n$. Then

$$\delta(h(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|h'(x_i)|} . \tag{E1}$$

Proof: Where f is any function of x, let

$$I \equiv \int_{-\infty}^{\infty} dx\, f(x)\, \delta(h(x)) \quad \text{and} \quad J \equiv \int_{-\infty}^{\infty} dx\, f(x) \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|h'(x_i)|} . \tag{E2}$$

The quantity J is easily evaluated as

$$J \equiv \sum_{i=1}^{n} \int_{-\infty}^{\infty} dx\, f(x)\, \frac{\delta(x - x_i)}{|h'(x_i)|} = \sum_{i=1}^{n} \frac{f(x_i)}{|h'(x_i)|} . \tag{E3}$$

To evaluate I, let the x-axis be partitioned by points $a_0, a_1, \ldots, a_n$, which satisfy $-\infty = a_0 < a_1 < \cdots < a_n = \infty$, and also $x_i \in (a_{i-1}, a_i)$. Then

$$I = \sum_{i=1}^{n} I_i , \tag{E4}$$

where

$$I_i \equiv \int_{a_{i-1}}^{a_i} dx\, f(x)\, \delta(h(x)) . \tag{E5a}$$

Now, since the argument of the delta function in Eq. (E5a) vanishes inside (a_{i-1}, a_i) only at $x = x_i$, then we can also write I_i as

$$I_i = \int_{b_{i-1}}^{b_i} dx\, f(x)\, \delta(h(x)) , \tag{E5b}$$

where b_{i-1} and b_i are any numbers satisfying $a_{i-1} \le b_{i-1} < x_i < b_i \le a_i$. Let us choose the interval (b_{i-1}, b_i) small enough so that h', which by hypothesis is nonzero at $x = x_i$, is nonzero everywhere in (b_{i-1}, b_i). In that case we can view the relation $\xi = h(x)$ as a one-to-one mapping of the interval (b_{i-1}, b_i) about $x = x_i$ onto some interval (c_{i-1}, c_i) about $\xi = 0$. It is then permissible to change the integration variable in Eq. (E5b) from x to $\xi = f(x)$ by putting $x = h^{-1}(\xi)$ and

$$dx = \frac{d\xi}{|d\xi/dx|} = \frac{d\xi}{|h'(x)|} = \frac{d\xi}{|h'(h^{-1}(\xi))|} .$$

Eq. (E5b) thus transforms to

$$I_i = \int_{c_{i-1}}^{c_i} \frac{d\xi}{|h'(h^{-1}(\xi))|} f(h^{-1}(\xi))\, \delta(\xi) = \frac{f(h^{-1}(0))}{|h'(h^{-1}(0))|} ,$$

or, since x_i is the only zero of h in (b_{i-1}, b_i),

$$I_i = \frac{f(x_i)}{|h'(x_i)|}. \tag{E5c}$$

Inserting Eq. (E5c) into Eq. (E4) and then comparing with Eq. (E3), we conclude that $I = J$. Therefore, we have shown that

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(h(x)) = \int_{-\infty}^{\infty} dx\, f(x) \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|h'(x_i)|} \tag{E6}$$

for any function $f(x)$. Now taking $f(x) = \delta(x - x')$, the x-integrations on both sides are trivially performed, and the result (apart from x being replaced by x') is Eq. (E1). QED

As an sample application of the foregoing theorem, suppose $h(x) = y - x^2$, where $y > 0$. Clearly, $h(x)$ vanishes at, and only at, the two points $x_1 = y^{1/2}$ and $x_2 = -y^{1/2}$. Also, since $h'(x) = -2x$, then $|h'(x_1)| = |h'(x_2)| = 2y^{1/2}$. Therefore, Eq. (E1) allows us to conclude that

$$\delta(y - x^2) = \theta(y)\, \frac{\delta(x - y^{1/2}) + \delta(x + y^{1/2})}{2y^{1/2}}, \tag{E7}$$

where the θ-function ensures that y is indeed non-negative (so that $y^{1/2}$ is real).

ADDENDUM F. An Application of RVT Corollary II When f Is Two-to-One.

Suppose we wish to calculate the density function Q of $Y = f(X) = X^2$ when X is uniformly distributed on $[-c_1, c_2]$, where c_1 and c_2 are both non-negative. Since in this case f is not strictly one-to-one (e.g., f sends both 2 and -2 into 4), then Eqs. (39) are not applicable. But one-to-oneness is *not* required of f by RVT Corollary II; thus, we can write from Eqs. (32) and (11),

$$Q(y) = \int_{-\infty}^{\infty} dx\, (c_1 + c_2)^{-1}\, \theta(x + c_1)\, \theta(c_2 - x)\, \delta(y - x^2). \tag{F1}$$

Now, there are two ways of evaluating this integral. One way is to first render it trivial by making the change of integration variable $x \to z = x^2$. The inverse of this transformation can be either $x = -z^{1/2}$ or $x = +z^{1/2}$, but this is *not* an ambiguity so far as the above integral is concerned; because, when that integral is written as a sum of two integrals, one over the negative x-axis and one over the positive x-axis,

$$Q(y) \doteq \int_{-\infty}^{0} dx\, (c_1 + c_2)^{-1}\, \theta(x + c_1)\, \delta(y - x^2) + \int_{0}^{\infty} dx\, (c_1 + c_2)^{-1}\, \theta(c_2 - x)\, \delta(y - x^2),$$

we have no choice but to take $x = -z^{1/2}$ in the first integral and $x = +z^{1/2}$ in the second. Thus,

$$Q(y) = \int_{+\infty}^{0} (-\tfrac{1}{2}z^{-1/2}dz)(c_1+c_2)^{-1}\,\theta(-z^{1/2}+c_1)\,\delta(y-z)$$

$$+ \int_{0}^{\infty} (+\tfrac{1}{2}z^{-1/2}dz)(c_1+c_2)^{-1}\,\theta(c_2-z^{1/2})\,\delta(y-z)$$

$$= [2(c_1+c_2)]^{-1}\int_{0}^{\infty} dz\, z^{-1/2}\,\delta(y-z)\,[\theta(c_1-z^{1/2}) + \theta(c_2-z^{1/2})]$$

$$= [2(c_1+c_2)]^{-1}\int_{-\infty}^{\infty} dz\,\theta(z)\,z^{-1/2}\,\delta(y-z)\,[\theta(c_1^2-z) + \theta(c_2^2-z)].$$

The z-integration is now trivially performed, and yields

$$Q(y) = \theta(y)\,[\theta(c_1^2-y) + \theta(c_2^2-y)]\,[2(c_1+c_2)]^{-1}\,y^{-1/2}. \qquad \text{(F2)}$$

Another way to evaluate the integral in Eq. (F1) is to subject the delta function there to the delta function change-of-variable theorem [see Addendum E]. Thus, using Eq. (E7), we get

$$Q(y) = \int_{-\infty}^{\infty} dx\,(c_1+c_2)^{-1}\,\theta(x+c_1)\,\theta(c_2-x)\,\theta(y)\left\{\frac{\delta(x-y^{1/2}) + \delta(x+y^{1/2})}{2y^{1/2}}\right\}$$

$$= \theta(y)\,[2(c_1+c_2)]^{-1}\,y^{-1/2}\left\{\int_{-\infty}^{\infty} dx\,\theta(x+c_1)\,\theta(c_2-x)\,\delta(x-y^{1/2})\right.$$

$$\left. + \int_{-\infty}^{\infty} dx\,\theta(x+c_1)\,\theta(c_2-x)\,\delta(x+y^{1/2})\right\}.$$

Performing the x-integration gives

$$Q(y) = \theta(y)\,[2(c_1+c_2)]^{-1}\,y^{-1/2}\,\{\theta(y^{1/2}+c_1)\,\theta(c_2-y^{1/2}) + \theta(-y^{1/2}+c_1)\,\theta(c_2+y^{1/2})\}.$$

Since $y > 0$, the quantity in braces reduces to $\{\theta(c_2^{\,2}-y) + \theta(c_1^{\,2}-y)\}$, and we again obtain Eq. (F2).

ADDENDUM G. The Gamma Distribution.

If the random variables $X_1, \ldots, X_n$ $(n \geq 1)$ are mutually independent and each X_i is $E(a)$, then the random variable Y defined by

$$Y = \sum_{i=1}^{n} X_i \tag{G1}$$

is said to have a *gamma* distribution with parameters a and n; we say that "Y is $\Gamma(a,n)$."

To calculate the density function of Y, we first observe from Eqs. (12) and (16) that the joint density function of $X_1, \ldots, X_n$ is

$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} \theta(x_i)\, a \exp(-ax_i).$$

Thus, by RVT Corollary I, the density function $Q(y)$ of Y is

$$Q(y) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \prod_{i=1}^{n} \theta(x_i)\, a \exp(-ax_i)\, \delta\left(y - \sum_{j=1}^{n} x_j\right)$$

$$= a^n \int_0^{\infty} dx_1 \cdots \int_0^{\infty} dx_{n-1} \exp\left(-a \sum_{i=1}^{n-1} x_i\right) \int_{-\infty}^{\infty} dx_n\, \theta(x_n) \exp(-ax_n)\, \delta\left(x_n - \left[y - \sum_{j=1}^{n-1} x_j\right]\right)$$

$$= \theta(y)\, a^n \int_0^{\infty} dx_1 \cdots \int_0^{\infty} dx_{n-1} \exp\left(-a \sum_{i=1}^{n-1} x_i\right) \theta\left(y - \sum_{j=1}^{n-1} x_j\right) \exp\left(-a\left[y - \sum_{j=1}^{n-1} x_j\right]\right)$$

$$Q(y) = \theta(y)\, a^n \exp(-ay) \int_0^{y} dx_1 \int_0^{y-x_1} dx_2 \int_0^{y-x_1-x_2} dx_3 \cdots \int_0^{y-x_1-\cdots-x_{n-2}} dx_{n-1}. \tag{G2}$$

[The function $\theta(y)$ was inserted when the delta function was integrated out because the argument of that delta function never vanishes if $y<0$.] Next we change integration variables from $(x_1, \ldots, x_{n-1})$ to $(u_1, \ldots, u_{n-1})$, where

$$\begin{aligned} u_1 &= y - x_1, \\ u_2 &= y - x_1 - x_2, \\ &\cdots \\ u_{n-1} &= y - x_1 - x_2 - \cdots - x_{n-1}. \end{aligned}$$

Since this is a linear transformation whose matrix is triangular with -1's along its main diagonal, then the absolute value of its Jacobian is unity; furthermore, since

$$u_j \equiv y - \sum_{i=1}^{j} x_i \equiv y - \sum_{i=1}^{j-1} x_i - x_j \equiv u_{j-1} - x_j$$

(with $u_0 \equiv y$), then as the old integration variable x_j in Eq. (G2) ranges over the integral $[0, u_{j-1}]$, the corresponding new integration variable u_j ranges over that same interval. Eq. (G2) therefore transforms to

$$Q(y) = \theta(y)\, a^n \exp(-ay) \int_0^y du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-3}} du_{n-2} \int_0^{u_{n-2}} du_{n-1} . \qquad \text{(G3)}$$

This multiple integral is easily evaluated: The u_{n-1}-integration gives $u^1_{n-2}/1!$ as the integrand for the u_{n-2}-integration, which in turn gives $u^2_{n-3}/2!$ as the integrand for the u_{n-3}-integration, ..., which in turn gives $u_1^{n-2}/(n-2)!$ as the integrand for the u_1-integration, which finally gives $y^{n-1}/(n-1)!$. We thus conclude that

$$Q(y) = \theta(y)\, a \exp(-ay) \frac{(ay)^{n-1}}{(n-1)!} \qquad [Y \text{ is } \Gamma(a,n)]. \qquad \text{(G4)}$$

Using the integral identity Eq. (53), it is easy to prove from Eqs. (10) and (G4) that the mean and variance of the distribution $\Gamma(a,n)$ are

$$\langle Y \rangle = n/a \quad \text{and} \quad \langle Y^2 \rangle - \langle Y \rangle^2 = n/a^2 \qquad [Y \text{ is } \Gamma(a,n)]. \qquad \text{(G5)}$$

It is obvious from the definition of the gamma distribution in Eq. (G1) that

$$\Gamma(a,1) = \mathrm{E}(a), \qquad \text{(G6a)}$$

which is also clear upon comparing Eq. (G4) with Eq. (12). However, it is a little surprising to discover, by comparing Eq. (G4) with Eq. (54), that

$$\Gamma(\tfrac{1}{2},n) = \chi^2(2n). \qquad \text{(G6b)}$$

The asymptotic form of $\Gamma(a,n)$ for $n \to \infty$ can be deduced most easily by using the Central Limit Theorem: If Y is $\Gamma(a,n)$, then from Eqs. (G1) and (22a) it follows that the random variable Z defined by

$$Z \equiv \frac{1}{n} Y \equiv \frac{1}{n} \sum_{i=1}^{n} X_i$$

can be regarded as the n-sample mean of a random variable X, which in turn is $\mathrm{E}(a)$. Since X in that case has mean a^{-1} and variance a^{-2}, then the Central Limit Theorem tells us that the distribution of Z tends to $\mathrm{N}(a^{-1}, a^{-2}/n)$ as n tends to ∞. Thus, by Eq. (35), the distribution of $Y = nZ$ tends to $\mathrm{N}(na^{-1}, n^2 a^{-2}/n)$, or

$$\Gamma(a,n) \to \mathrm{N}(n/a,\, n/a^2) \quad \text{as} \quad n \to \infty. \qquad \text{(G7)}$$

The gamma distribution plays an obvious role in the mathematics of radioactive decay: Thus, for a radioactive sample whose decay constant is r decays per unit time, the *time* required for exactly

n decays is a random variable whose distribution is $\Gamma(r, n)$. Eq. (G7) would then imply that the time required for, say, 100 decays would be approximately normally distributed with mean $100/r$ and standard deviation $(100/r^2)^{1/2} = 10/r$. Similarly, in a dilute gas with mean-free-path λ, the total distance travelled by a molecule in making exactly n collisions is a random variable whose distribution is $\Gamma(\lambda^{-1}, n)$.

ADDENDUM H. Sum of Independent Normal Random Variables.

Let $X_1, \ldots, X_n$ be n mutually independent random variables, and let X_i be $N(\mu_i, \sigma_i^2)$ $(i = 1,\ldots,n)$. Then by RVT Corollary I, the density function $Q(y)$ of the random variable

$$Y = \sum_{i=1}^{n} X_i \tag{H1}$$

is

$$Q(y) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \prod_{i=1}^{n} (2\pi\sigma_i^2)^{-1/2} \exp(-[x_i - \mu_i]^2/2\sigma_i^2)\, \delta\!\left(y - \sum_{j=1}^{n} x_j\right). \tag{H2}$$

Using Eq. (2) the delta function here can be written

$$\delta\!\left(y - \sum_{j=1}^{n} x_j\right) = (2\pi)^{-1} \int_{-\infty}^{\infty} ds \exp(isy) \prod_{j=1}^{n} \exp(-isx_j).$$

With this, Eq. (H2) becomes

$$Q(y) = (2\pi)^{-1} \int_{-\infty}^{\infty} ds \exp(isy) \prod_{j=1}^{n} \int_{-\infty}^{\infty} dx_j\, (2\pi\sigma_j^2)^{-1/2} \exp(-isx_j) \exp(-[x_j - \mu_j]^2/2\sigma_j^2).$$

Changing integration variables from x_j to $z_j = (x_j - \mu_j)/\sigma_j$ gives

$$Q(y) = (2\pi)^{-1} \int_{-\infty}^{\infty} ds \exp(isy) \prod_{j=1}^{n} (2\pi)^{-1/2} \exp(-is\mu_j) \int_{-\infty}^{\infty} dz_j \exp(-is\sigma_j z_j) \exp(-z_j^2/2).$$

But

$$\int_{-\infty}^{\infty} dz_j \exp(-is\sigma_j z_j) \exp(-z_j^2/2) = 2 \int_0^{\infty} \cos(s\sigma_j z_j) \exp(-z_j^2/2)\, dz_j = (2\pi)^{1/2} \exp(-s^2\sigma_j^2/2),$$

where the last step follows from the integral identity Eq. (48). Thus,

$$Q(y) = (2\pi)^{-1} \int_{-\infty}^{\infty} ds \exp(isy) \prod_{j=1}^{n} \exp(-is\mu_j) \exp(-s^2\sigma_j^2/2).$$

Now defining

$$\mu_0 = \sum_{j=1}^{n} \mu_j \quad \text{and} \quad \sigma_0^2 = \sum_{j=1}^{n} \sigma_j^2, \tag{H3}$$

the above simplifies to

$$Q(y) = (2\pi)^{-1} \int_{-\infty}^{\infty} ds \exp(isy) \exp(-is\mu_0) \exp(-s^2\sigma_0^2/2)$$

$$= (2\pi)^{-1} 2 \int_0^{\infty} \cos(s[y-\mu_0]) \exp(-s^2\sigma_0^2/2)\, ds .$$

Appealing once more to the integral identity Eq. (49), we conclude that

$$Q(y) = (2\pi\sigma_0^2)^{-1/2} \exp(-[y-\mu_0]^2/2\sigma_0^2) . \tag{H4}$$

We have thus proved that the sum of n independent normally distributed random variables is itself normally distributed, with mean and variance equal, respectively, to the sums of the means and variances of the given random variables.

The foregoing result tells us that if X_1 is $N(\mu_1, \sigma_1^2)$ and X_2 is $N(\mu_2, \sigma_2^2)$, with X_1 and X_2 mutually independent, then the sum $X_1 + X_2$ is $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. The *difference* $X_1 - X_2$ is handled by regarding it as the sum of X_1 and $-X_2$: According to Eq. (36), $-X_2$ is $N((-1)\mu_2, (-1)^2\sigma_2^2) = N(-\mu_2, \sigma_2^2)$; thus, $X_1 - X_2$ is $N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$.

If X_i is regarded as the i^{th} sampling of X [see Eq. (21)], where X is $N(\mu, \sigma^2)$, then we have $\mu_i = \mu$ and $\sigma_i = \sigma$ for all $i = 1,\ldots,n$, and the foregoing result implies that

$$\sum_{i=1}^{n} X_i \quad \text{is} \quad N(n\mu, n\sigma^2).$$

It then follows from Eqs. (22a) and (35) that, if X is $N(\mu, \sigma^2)$, then $\bar{X}_n$ is $N(n^{-1}n\mu, n^{-2}n\sigma^2) = N(\mu, \sigma^2/n)$. This constitutes an independent derivation of the important result in Eq. (77a).

ADDENDUM I. Orthogonal Transformations

Following the classical exposition of Goldstein [see Ref. 7], we review here some properties of orthogonal transformations that are invoked in Applications 8 and 10.

The linear transformation from $z_1 z_2 \cdots z_n$-space to $u_1 u_2 \cdots u_n$-space,

$$u_i = \sum_{j=1}^{n} a_{ij} z_j \qquad (i = 1,\ldots,n), \tag{I1}$$

is said to be *orthogonal* if and only if

$$\sum_{i=1}^{n} z_i^2 = \sum_{i=1}^{n} u_i^2. \qquad \text{(I2)}$$

Substituting Eq. (I1) into the right side of Eq. (I2) gives

$$\sum_{i=1}^{n} z_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij}a_{ik}z_j z_k ,$$

from which it is seen that Eq. (I2) holds if and only if

$$\sum_{i=1}^{n} a_{ij}a_{ik} = \delta_{jk} \quad (j,k=1,\ldots,n), \qquad \text{(I3)}$$

where δ_{jk} is the Kronecker delta symbol. This is one form of the orthogonality condition, although it is not the form that we require in Applications 8 and 10.

Let the transformation *inverse* to Eq. (I1) be

$$z_j = \sum_{k=1}^{n} a'_{jk}u_k \quad (j=1,\ldots,n). \qquad \text{(I4)}$$

Inserting this into Eq. (I1) gives

$$u_i = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij}a'_{jk}u_k ,$$

which can evidently hold for all u_i only if

$$\sum_{j=1}^{n} a_{ij}a'_{jk} = \delta_{ik} \quad (i,k=1,\ldots,n). \qquad \text{(I5)}$$

Now consider the double sum,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ik}a_{ij}a'_{jl} .$$

If we evaluate this by summing first over i with the help of Eq. (I3), the result is a'_{kl}. But if we sum first over j with the help of Eq. (I5), the result is a_{lk}. We conclude that

$$a'_{kl} = a_{lk} \quad (k,l=1,\ldots,n). \qquad \text{(I6)}$$

Substituting this into Eq. (I5) gives

$$\sum_{j=1}^{n} a_{ij}a_{kj} = \delta_{ik} \quad (i,k=1,\ldots,n), \qquad \text{(I7)}$$

which is the form of the orthogonality condition invoked in Eqs. (69) and (107a).

The Jacobian of the transformation in Eq. (I1) is clearly equal to the determinant of the matrix **A**, where $(\mathbf{A})_{ij} \equiv a_{ij}$:

$$\frac{\partial(u_1, \dots, u_n)}{\partial(z_1, \dots, z_n)} = |\mathbf{A}|. \tag{I8}$$

Denoting the inverse of **A** by $\mathbf{A}^{-1}$ and the transpose of **A** by $\mathbf{A}^{\mathrm{T}}$, we have

$$1 = |\mathbf{A}\,\mathbf{A}^{-1}| = |\mathbf{A}\,\mathbf{A}^{\mathrm{T}}| = |\mathbf{A}|\,|\mathbf{A}^{\mathrm{T}}| = |\mathbf{A}|^2 . \tag{I9}$$

Here, the second equality follows from Eq. (I6), the third equality follows from the fact that the multiplication rule for matrices is the same as the multiplication rule for determinants, and the last equality follows from the fact that transposition does not alter the value of a determinant. Combining Eqs. (I8) and (I9) gives

$$\left| \frac{\partial(u_1, \dots, u_n)}{\partial(z_1, \dots, z_n)} \right| = 1. \tag{I10}$$

Eq. (I2) says that orthogonal transformations preserve *length*, and Eq. (I10) says that orthogonal transformations preserve *volume*.

ADDENDUM J. Asymptotic Equivalence of the Poisson and Normal Distributions

Using an argument similar to that given by Present [see Ref. 9], we show here that the Poisson distribution with a large mean can be approximated by the normal distribution that has the same mean and variance. This fact was invoked in our proof of the chi-square theorem [cf. Eqs. (95) and (96)].

The discrete-variable Poisson probability function with mean and variance a is

$$P(x) = a^x e^{-a}/x! \qquad (x = 0, 1, \dots). \tag{J1}$$

We consider here only the circumstance

$$a \gg 1, \tag{J2}$$

in which case $P(x)$ differs appreciably from zero only if x satisfies

$$|x - a| \ll a. \tag{J3}$$

If a satisfies Eq. (J2), then any x satisfying Eq. (J3) will be very large compared to unity; we can therefore invoke Stirling's formula,

$$x! \approx (2\pi x)^{1/2} x^x e^{-x} \quad (x \gg 1), \tag{J4}$$

and so approximate Eq. (J1) by

$$P(x) \approx (2\pi x)^{-1/2}(a/x)^x e^{x-a}$$
$$= (2\pi a)^{-1/2}(x/a)^{-(x+1/2)} e^{x-a}$$
$$\approx (2\pi a)^{-1/2}(x/a)^{-x} e^{x-a},$$

where the last step follows since $x \gg 1$. Rearranging and taking logarithms gives

$$\log[(2\pi a)^{1/2} P(x)] \approx -x \log(x/a) + (x-a),$$

or, defining $\varepsilon \equiv x-a$,

$$\log[(2\pi a)^{1/2} P(x)] \approx -(\varepsilon+a)\log[1+(\varepsilon/a)] + \varepsilon .$$

Eq. (J3) implies that $|\varepsilon/a| \ll 1$, so we can expand the logarithm on the right and retain only the lowest order terms in ε/a:

$$\log[(2\pi a)^{1/2} P(x)] \approx -(\varepsilon+a)[(\varepsilon/a) - \tfrac{1}{2}(\varepsilon/a)^2 + \cdots] + \varepsilon$$
$$\approx -\tfrac{1}{2}(\varepsilon^2/a) \equiv -(x-a)^2/2a.$$

We therefore conclude that, when $a \gg 1$, the Poisson function in Eq. (J1) can be approximated by

$$P(x) \approx (2\pi a)^{-1/2}\exp[-(x-a)^2/2a] \qquad \text{(J5)}$$

in the region where it differs appreciably from zero. The quantity on the right of Eq. (J5) is of course the density function of the normal distribution with mean and variance a.

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